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Intertwined Recursion, Tree Transformations, and Linear Systems*

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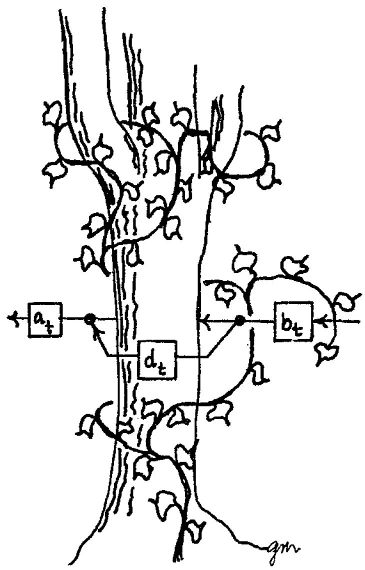

Motivated by the way in which the recursive definition of the response of a generalized sequential machine is intertwined with that of the reachability map, we introduce an *intertwined recursion principle* valid for any endofunctor that admits free dynamics. This allows us to extend the Arbib–Manes definition of a machine in a category to that of a *process transformation* which transforms input processes to output processes. This formalization includes primitive recursion, generalized sequential machines, bottom-up tree transformations, and a generalized notion of linear systems which treats the initial state and input on a symmetric footing in its reachability theory. We also analyze the behavior of loop-free networks of process transformations.

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1. Introduction

There are two main approaches to the category-theoretic formulation of systems. The *closed-category approach* (see, e.g., Goguen, 1972; Ehrig *et al.*, 1974) takes as its setting a closed category $\mathscr{K}$ with denumerable coproducts, and takes the state-space Q, output-space Y, and input-space X_0 of a system to be objects of $\mathscr{K}$. A *dynamics* is then a morphism,

$$\delta : Q \otimes X_0 \to Y, \tag{1}$$

while the *output map* of the system is another morphism,

$$\lambda : Q \otimes X_0 \to Y. \tag{2}$$

The advantage of this approach is that we can readily define X_0^* and Y_0^* as the coproduct of n-fold tensor products, $n \geqslant 0$, of X_0 and Y_0, respectively, and then extend (δ, λ) to a *response morphism* $X_0^* \to Y_0^*$. The disadvantage of this approach is its limited applicability: It includes sequential machines and bilinear machines, but does not include linear systems and tree automata.

The *recursion-process approach*[1] (see, e.g., Arbib and Manes, 1974a; see Bainbridge, 1973, for a related approach) takes as its setting any category $\mathscr{K}$, takes the state-space Q and output-space Y to be objects of $\mathscr{K}$, and takes the input X to be a *functor* $X: \mathscr{K} \to \mathscr{K}$ which is a recursion process in the sense of Definition 2 of Section 2 below. A *dynamics* is then a morphism,

$$\delta : QX \to Q, \tag{3}$$

while the *output map* of the system is another morphism,

$$\beta : Q \to Y. \tag{4}$$

An initial state map $\tau : A \to Q$ extends[2] to a reachability map $r: AX^{@} \to Q$ (and, taking $X = - \otimes X_0$ in a suitable $\mathscr{K}$, this includes the definition $r: A \otimes X_0^* \to Q$ of the closed-category approach). However, the disadvantage of this approach is its asymmetry of treatment of input and output—Y has no analog of Y_0^* in the way that $X^{@}$ provides an analog of X_0^*. In particular, we have no definition of a response map of a form $AX^{@} \to BY^{@}$ for a system represented by (3) and (4). However, the advantages of the approach are considerable. It not only handles sequential machines and bilinear machines, but also includes linear machines, tree automata, and many others (see, e.g., Arbib and Manes, 1974a). Can we, then, preserve these advantages yet also provide the analog of the response map $X_0^* \to Y_0^*$? Our observation above that a

[1] Elsewhere called the input-process approach. The reason for the renaming is given at the start of Section 2.

[2] We explain how—and define $X^{@}$—in Definition 2.2.

suitable analog might be of the form $AX^{@} \to BY^{@}$ provides the key to the answer—input and output must be treated on an equal footing, with both X and Y being recursion processes. An elegant analysis along these lines was provided by Alagić (1975), who, motivated by the way in which the dynamics and output map of a generalized sequential machine are captured in a single map,

$$Q \times X_0 \to Y_0^* \times Q, \tag{5}$$

offered the general concept of a *direct state transformation* which took the form[3] of a natural transformation,[4]

$$\bar{Q}X \to Y^{@}\bar{Q}, \tag{6}$$

where X and Y are recursion processes and $\bar{Q}$ is now a functor. A major motivation for Alagić's paper was the study of tree transformations, and he showed that (6) subsumed the bottom-up tree transformations of Engelfriet (1975) and the generalized[2] sequential machines of Thatcher (1970). Alagić also defined *inverse state transformations* to be natural transformations of the form

$$X\bar{Q} \to \bar{Q}Y^{@}, \tag{7}$$

where X and Y are again recursion processes, and the state-functor $\bar{Q}$ is now required to have a right adjoint. Alagić shows that this notion subsumes top-down tree transformations. He proves a number of interesting results about these transformations, including the result (stated at the top of p. 299 of his paper as part of his proof of Theorem 3.10) that to every inverse state transformation on a free monad there corresponds a pure direct state transformation on a free monad [the reader is referred to Alagić (1975) for the definitions of this terminology].

But the Alagić approach has one flaw: Because $\bar{Q}$ is a functor rather than an object, the state is "entangled" with the input and output, so that "running" the direct state transformation (6) yields

$$\bar{Q}X^{@} \to Y^{@}\bar{Q}, \tag{8}$$

but there seems no general way to introduce objects A and B in such a way that we can extract from (8) a "state-free" input–output response,

$$AX^{@} \to BY^{@}, \tag{9}$$

as a suitable generalization of the $f = \beta \cdot r \colon AX^{@} \to Y$ available for machines described by (3) and (4). Our major contribution, then, is to show that the benefits of the Alagić approach can be obtained in any category with binary

[3] More generally, Alagić replaced $Y^{@}$ by the T of any algebraic theory.

[4] The reader unfamiliar with natural transformations will find an exposition in Section 3 below.

products, and that we can once more use a state-*object* Q, with Alagić's state-functor $\bar{Q}$ restricted to the special form $\hat{Q} = -\times Q$. In this case, the direct state transformation $\hat{Q}X \to Y^{@}\hat{Q}$ unpacks into a dynamics $QX \to Q$ together with a natural transformation $\hat{Q}X \to Y^{@}$. These two maps are at the heart of the notion of a process transformation which we develop in this paper. While these two maps may be seen as a specialization of Alagić's machinery, the research reported here required delicate analysis to reveal the proper way of handling A and B to yield a response of the form (9). Our development is based on an *intertwined recursion principle* which makes explicit how the definition of the response (9) of a process transformation is intertwined with the definition of an appropriate reachability map $r: AX^{@} \to Q$. We show that our notion of a process transformation not only covers all the specific applications which Alagić provided for his direct state transformations, but also includes primitive recursion, and provides an insightful analysis of linear systems which shows that input and initial state may be treated on a surprisingly symmetric basis when considering reachability, but that this symmetry is lost when we consider the response $AX^{@} \to BY^{@}$.

Apart from some basic familiarity with the notion of a recursion process and the necessary elements of category theory (see, e.g., Arbib and Manes, 1975a), the paper is self-contained. In particular, no use is made of the results from Alagić (1975). Where Alagić offers an analysis of serial composition of state transformations, we offer an analysis of cascade connection of process transformations, which includes both serial and parallel connections.

2. The Intertwined Recursion Principle

In earlier papers (see, e.g., Arbib and Manes, 1974a) we have studied the category $\mathrm{Dyn}(X)$ of X-dynamics for endofunctors $X: \mathscr{K} \to \mathscr{K}$, and seen that "running a dynamics" corresponds to X being a *recursion process*. (We have used the term *input process* in earlier papers, but abandon it now since, in this paper, we consider systems whose outputs, as well as inputs, are recursion processes.)

1. Definition. Let $X: \mathscr{K} \to \mathscr{K}$ be any endofunctor. An *X-dynamics* is a pair (Q, δ) where Q is an object and $\delta: QX \to Q$ is a morphism in $\mathscr{K}$. Given two X-dynamics (Q, δ), (Q', δ'), a morphism $h: Q \to Q'$ is an *X-dynamorphism* if

$$\begin{tikzcd} QX \arrow[r, "\delta"] \arrow[d, "hX"'] & Q \arrow[d, "h"] \\ Q'X \arrow[r, "\delta'"] & Q' \end{tikzcd}$$

We obtain a category $\mathrm{Dyn}(X)$ with composition and identities at the level of $\mathscr{K}$.

2. Definition. We say that $X\colon \mathscr{K} \to \mathscr{K}$ is a *recursion process* if there exists a *free dynamics* $(AX^@, A\mu_0)$ over each object A in $\mathscr{K}$; i.e., $(AX^@, A\mu_0)$ is coupled with a morphism $A\eta\colon A \to AX^@$ with the universal property that for every other pair of an X-dynamics (Q, δ) and morphism $\tau\colon A \to Q$ there exists a unique X-dynamorphism $r\colon (AX^@, A\mu_0) \to (Q, \delta)$ such that $r \cdot A\eta = \tau$. That is, given τ and δ

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\tau"'] & AX^@ \arrow[d, dashed, "r"] & AX^@X \arrow[l, "A\mu_0"'] \arrow[d, dashed, "rX"] \\ & Q & QX \arrow[l, "\delta"] \end{tikzcd} \tag{3}$$

there exists a unique r such that (3) commutes.

It can be easily shown that $X^@$ in (3) is the object map of a functor $X^@\colon \mathscr{K} \to \mathscr{K}$. Additionally, we note that each recursion process yields a family of maps

$$A\mu\colon AX^@X^@ \to AX^@$$

defined by the diagram

$$\begin{tikzcd} AX^@ \arrow[r, "AX^@\eta"] \arrow[dr, "\mathrm{id}_{AX^@}"'] & AX^@X^@ \arrow[d, dashed, "A\mu"] & AX^@X^@X \arrow[l, "AX^@\mu_0"'] \arrow[d, dashed, "A\mu X"] \\ & AX^@ & AX^@X \arrow[l, "A\mu_0"] \end{tikzcd} \tag{4}$$

We now show that (3) includes the classical scheme of simple recursion. Let $\mathbf{N}$ be the set of natural numbers, let A, B be sets, and let $\alpha\colon A \to B$, $\Gamma\colon B \to B$ be maps. Then the scheme

$$\gamma(a, 0) = \alpha(a)$$

$$\gamma(a, n + 1) = \Gamma(\gamma(a, n))$$

defines a unique function $\gamma\colon A \times \mathbf{N} \to B$. We say that γ is defined by *simple recursion* from α and Γ. Now this yields the diagram

$$\begin{tikzcd} A \arrow[r, "0_A"] \arrow[dr, "\alpha"'] & A \times \mathbf{N} \arrow[d, "\gamma"] & A \times \mathbf{N} \arrow[l, "\mathrm{id}_A \times s"'] \arrow[d, "\gamma"] \\ & B & B \arrow[l, "\Gamma"] \end{tikzcd} \tag{5}$$

where $s\colon \mathbf{N} \to \mathbf{N}$, $n \mapsto n + 1$ is the successor function, $0_A : A \to A \times \mathbf{N}$, $a \mapsto (a, 0)$ is the zero function. That is, given α and Γ, there is a unique γ such that (5) commutes. [It is well known (Lawvere, 1964; Freyd, 1972, Proposition 5.22) that any natural numbers object $\mathbf{N}$ in a topos satisfies the property.] We now observe that (5) is the special case of (3) obtained by setting

$$\mathscr{K} = \text{Set}, \qquad X = \mathrm{id}_{\text{Set}},$$

where we then have that

$$AX^{@} = A \times \mathbf{N}, \qquad A\mu_0 = \mathrm{id}_A \times s, \qquad A\eta = 0_A.$$

We now turn to our version of Mealy sequential machines, and see that the definition of the reachability map is again an instance of (3), but that the definition of the response map requires an extension of (3) which—motivated by the above discussion of the simple recursion recursion principle—we call the intertwined recursion principle.

6. Definition. Given sets A, B, X_0, Y_0 a *Mealy sequential machine* M: $(A, X_0) \to (B, Y_0)$

$$(A, X_0) \longrightarrow \boxed{M} \longrightarrow (B, Y_0)$$

is a quadruple $M = (Q, \delta, \tau, \alpha, \lambda)$ with

$$\begin{aligned} &\delta: Q \times X_0 \to Q, \\ &\tau: A \to Q, \\ &\alpha: A \to B, \\ &\lambda: Q \times X_0 \to Y_0. \end{aligned}$$

By a *generalized sequential machine* $M: (A, X_0) \to (B, Y_0)$ we mean $M = (Q, \delta, \tau, \alpha, \lambda)$ with δ, τ as above but with

$$\begin{aligned} &\alpha: A \to B \times Y_0^*, \\ &\lambda: Q \times X_0 \to Y_0^*, \end{aligned}$$

where Y_0^* is the free monoid generated by Y_0. Injecting B into $B \times Y^*$ using the empty string and regarding Y_0 as a subset of Y_0^*, each Mealy sequential machine is also a generalized sequential machine.

Before going further, we must note that our definition of a Mealy machine generalizes the usual definition by requiring a *set* A of initial state labels together with an initial state map $\tau: A \to Q$. The usual definition assumes that A has only one element, a say, and simply lists $\tau(a) = q_0$ as *the* initial state. However, the use of multiple initial states is crucial in the theory of tree automata, for example, and so we introduce them here to aid the reader in building up intuition for our general definition, 3.12. We also see a crucial role for nontrivial A in our study of linear systems in Section 6. Having accepted the use of a set A of initial state labels, an experiment with a Mealy machine consists of specifying an element (a, w) of $A \times X^*$—initializing the state of the machine to $\tau(a)$, and then observing the response to the input string w. We must define the observable output to belong to a structure similar to $A \times X_0^*$—and we do this by introducing the set B, and taking the response of the machine to lie in $B \times Y_0^*$ (and thus able to serve as input to yet another transducer). The *initial throughput map* α provides the basis step for defining the response map $\gamma: A \times X_0^* \to B \times Y_0^*$; the induction steps then require "intertwined" use of δ and λ as we see in the next definition. Of course, in the familiar case where A and B have only one element, α becomes superfluous and we may view γ as a map $X_0^* \to Y_0^*$.

7. Definition. Let $M = (Q, \delta, \tau, \alpha, \lambda): (A, X_0) \to (B, Y_0)$ be a generalized sequential machine. Then the *reachability map* $r: A \times X_0^* \to Q$ is defined by

$$\begin{array}{lll} \text{Basis Step:} & r(a, \Lambda) = \tau(a) & (\Lambda \text{ the empty word}), \\ \text{Induction Step:} & r(a, wx) = \delta(r(a, w), x) & (w \in X_0^*,\ x \in X_0). \end{array} \tag{8}$$

The *response map* $\gamma: A \times X^* \to B \times Y^*$ is defined by

$$\begin{array}{lll} \text{Basis Step:} & \gamma(a, \Lambda) = \alpha(a), & \\ \text{Induction Step:} & \gamma(a, wx) = \gamma(a, w) \cdot \lambda(r(a, w), x) & (w \in X_0^*,\ x \in X_0), \end{array} \tag{9}$$

where we use the notation $(b, w) \cdot v$ for (b, wv).

We see that (8) may be rewritten

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\tau"'] & A \times X_0^* \arrow[d, "r"] & A \times X_0^* \times X_0 \arrow[l, "A\mu_0"'] \arrow[d, "r \times X_0"] \\ & Q & Q \times X_0 \arrow[l, "\delta"] \end{tikzcd} \tag{10}$$

which is clearly the special case of (3) obtained by taking

$$\mathscr{K} = \text{Set}, \qquad X = - \times X_0,$$

where we then have that

$$AX^{@} = A \times X_0^*, \qquad A\mu_0(a, w, x) = (a, wx), \qquad A\eta(a) = (a, \Lambda).$$

Incidentally, note that in this case the concatenation $A \times X_0^* \times X_0^* \to A \times X_0^*$, $(a, w, v) \mapsto (a, w \cdot v)$ is just the $A\mu: AX^{@}X^{@} \to AX^{@}$ of (4).

Now, (9) requires a "recursion" that is "intertwined" in the sense that the induction step requires that the previous step of r, as well as that of γ, be available. Diagrammatically, (9) becomes

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\alpha"'] & A \times X_0^* \arrow[d, "\gamma"] & A \times X_0^* \times X_0 \arrow[l, "A\mu_0"'] \arrow[d, "\binom{\gamma}{r} \times X_0"] \\ & B \times Y_0^* & (B \times Y_0^* \times Q) \times X_0 \arrow[l, "\Gamma"] \end{tikzcd} \tag{11}$$

where $\binom{\gamma}{r}: A \times Y_0^* \to B \times Y_0^* \times Q: (a, w) \mapsto (\gamma(a, w), r(a, w))$, and $\Gamma(b, v, q, x) = (b, v \cdot \lambda(q, x))$ so that the square says

$$\gamma(a, wx) = \Gamma(\gamma(a, w), r(a, w), x) = \gamma(a, w) \cdot \lambda(r(a, w), x).$$

Just as (10) was a special case of (3), so may we see that (11) is a special case of (13) below:

12. The Intertwined Recursion Principle. Let $\mathscr{K}$ be a category with binary products, and let $X: \mathscr{K} \to \mathscr{K}$ be a recursion process. Then, given $\tau: A \to Q$, $\delta: QX \to Q$, $\alpha: A \to K$, and $\Gamma: (K \times Q)X \to K$, there exists a unique $\gamma: AX^@ \to K$ such that, with the $r: AX^@ \to Q$ defined by τ and δ as in (3) we have

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\alpha"'] & AX^@ \arrow[d, "\gamma"] & AX^@X \arrow[l, "A\mu_0"'] \arrow[d, "\binom{\gamma}{r}X"] \\ & K & (K \times Q)X \arrow[l, "\Gamma"] \end{tikzcd} \tag{13}$$

We say that γ *is defined from* α *and* Γ *by intertwined recursion with* r.

Proof. Given Γ and δ we may define the X-dynamics

$$\binom{\Gamma}{\delta \cdot \mathrm{pr}_2 X}: (K \times Q)X \to K \times Q$$

which then lets us apply (3) in the form

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\binom{\alpha}{\tau}"'] & AX^@ \arrow[d, dashed, "\binom{\gamma}{\bar{r}}"] & AX^@X \arrow[l, "A\mu_0"'] \arrow[d, "\binom{\gamma}{\bar{r}}X"] \\ & K \times Q & (K \times Q)X \arrow[l, "\binom{\Gamma}{\delta \cdot \mathrm{pr}_2 X}"] \end{tikzcd} \tag{14}$$

to develop a unique pair $(\gamma: AX^@ \to K, \bar{r}: AX^@ \to Q)$. Via the projections $Q \leftarrow Q \times K \to K$, (14) is equivalent to (15) and (16):

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\tau"'] & AX^@ \arrow[d, "\bar{r}"] & AX^@X \arrow[l, "A\mu_0"'] \arrow[d, "\bar{r}X"] \\ & Q & QX \arrow[l, "\delta"] \end{tikzcd} \tag{15}$$

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\alpha"'] & AX^@ \arrow[d, "\gamma"] & AX^@X \arrow[l, "A\mu_0"'] \arrow[d, "\binom{\gamma}{\bar{r}}X"] \\ & K & (K \times Q)X \arrow[l, "\Gamma"] \end{tikzcd} \tag{16}$$

Comparing (15) and (3) we see that, by uniqueness, $r = \bar{r}$ so that (16) is just (13). Now if γ' also satisfies (13), we have that (15) and (16) hold with $\bar{r} = r$, $\gamma = \gamma'$, so that (14) holds, yielding $\binom{\gamma}{r} = \binom{\gamma'}{r}$ and hence $\gamma = \gamma'$. ∎

Just as we saw that simple recursion (5) was a special case of the recursion process setting (3), so we now see that the classical notion of primitive recursion is a special case of intertwined recursion of (13). Given $\alpha\colon A \to K$ and $\Gamma\colon K \times A \times \mathbf{N} \to K$, we say that $\gamma\colon A \times \mathbf{N} \to K$ is obtained from α and γ by *primitive recursion* if it is defined by

$$\gamma(a, 0) = \alpha(a),$$
$$\gamma(a, n+1) = \Gamma(\gamma(a, n), a, n).$$

But this is equivalent to the diagram

$$\begin{tikzcd} A \arrow[r, "0_A"] \arrow[dr, "\alpha"'] & A \times \mathbf{N} \arrow[d, "\gamma"] & & A \times \mathbf{N} \arrow[ll, "\mathrm{id}_A \times s"'] \arrow[d, "\binom{\gamma}{\mathrm{id}_{A\times\mathbf{N}}}"] \\ & K & & K \times (A \times \mathbf{N}) \arrow[ll, "\Gamma"'] \end{tikzcd} \tag{17}$$

which corresponds to (13) with $\mathscr{K} = \mathrm{Set}$, $X = \mathrm{id}_{\mathrm{Set}}$, where we then have, as in (5), that

$$AX^{@} = A \times \mathbf{N}, \qquad A\eta = 0_A, \qquad A\mu_0 = \mathrm{id}_A \times s.$$

Finally, we take our underlying dynamics to be the free dynamics over A, i.e.,

$$Q = A \times \mathbf{N}, \qquad \text{with } \tau = 0_A : A \to A \times \mathbf{N},\ \delta = \mathrm{id}_A \times s$$

which has reachability map $r = \mathrm{id}_{A\times\mathbf{N}}$.

3. Process Transformations

In 2.12, we established the intertwined recursion principle, namely, that to each $\tau\colon A \to Q$, $\delta\colon QX \to Q$, $\alpha\colon A \to K$, and $\Gamma\colon (K \times Q)X \to K$ we can assign a unique "response" $\gamma\colon AX^{@} \to K$

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\alpha"'] & AX^{@} \arrow[d, "\gamma"] & AX^{@}X \arrow[l, "A\mu_0"'] \arrow[d, "\binom{\gamma}{r}X"] \\ & K & (K \times Q)X \arrow[l, "\Gamma"'] \end{tikzcd} \tag{1}$$

where $r\colon AX^{@} \to Q$ is the reachability map of (τ, δ). As a special case of this, we saw in 2.11 that we had the response of a generalized sequential machine

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\alpha"'] & A \times X_0^* \arrow[d, "\gamma"] & A \times X_0^* \times X_0 \arrow[l, "A\mu_0"'] \arrow[d, "\binom{\gamma}{r}\times X_0"] \\ & B \times Y_0^* & (B \times Y_0^* \times Q) \times X_0 \arrow[l, "\Gamma"'] \end{tikzcd} \tag{2}$$

where Γ now takes the special form

$$B \times Y_0^* \times Q \times X_0 \xrightarrow{B\times Y_0^*\times\lambda} B \times Y_0^* \times Y_0^* \xrightarrow{B\times\text{concatenation}} B \times Y_0^*.$$

If we introduce the functors $\hat{Q} = - \times Q$, $X = - \times X_0$, and $Y = - \times Y_0$, this takes the form (recall 2.4 and the comment following 2.10)

$$BY^{@}\hat{Q}X \xrightarrow{BY^{@}\beta} BY^{@}Y^{@} \xrightarrow{B\mu} BY^{@}, \tag{3}$$

where $\beta\colon \hat{Q}X \to Y^{@}$ is the natural transformation given by $K\beta\colon K\hat{Q}X \to KY^{@}\colon (k, q, x) \mapsto (k, \lambda(q, x))$. This immediately suggests the notion of *process transformation* given in (12) below as the appropriate categorical generalization of a generalized sequential machine. However, for completeness, we first give a brief treatment of natural transformations, and of functors of the form $\hat{Q} = - \times Q$.

4. Definition. A *natural transformation* $\Gamma\colon F \to G$ of functors $F, G\colon \mathscr{A} \to \mathscr{B}$ is an assignment of a $\mathscr{B}$-morphism $A\Gamma\colon AF \to AG$ for each object A of $\mathscr{A}$ in such a way that for each $\mathscr{A}$-morphism $f\colon A \to A'$ the square in (5)

$$\begin{tikzcd} A \arrow[d, "f"'] & AF \arrow[r, "A\Gamma"] \arrow[d, "fF"'] & AG \arrow[d, "fG"] \\ A' & A'F \arrow[r, "A'\Gamma"] & A'G \end{tikzcd} \tag{5}$$

commutes. Each such square is called a *naturality square*.

As an important example of natural transformations, we state, without proof, the following well-known fact:

6. Fact. If we fix a choice of $A \xrightarrow{A\eta} AX^{@}$ and $AX^{@}X \xrightarrow{A\mu_0} AX^{@}$ in 2.3 for each object A in $\mathscr{K}$, we obtain a pair of natural transformations

$$\eta\colon \mathrm{id}_{\mathscr{K}} \to X^{@},$$
$$\mu_0 : X^{@}X \to X^{@}.$$

Moreover, the $A\mu\colon AX^{@}X^{@} \to AX^{@}$ of 2.4 define a natural transformation

$$\mu\colon X^{@}X^{@} \to X^{@}. \quad \blacksquare$$

7. Objects as Functors. Let $\mathscr{K}$ be a category with binary products and a terminal object 1. Given $f: A \to B$, $g: A' \to B'$, we define $f \times g: A \times A' \to B \times B'$ by

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times A' \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "f\times g"] & A' \arrow[d, "g"] \\ B & B \times B' \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & B' \end{tikzcd}$$

Relative to our choice of the assignment of products to pairs of objects, each object Q of $\mathscr{K}$ induces a functor $\hat{Q}: \mathscr{K} \to \mathscr{K}$ by $A\hat{Q} = A \times Q, f\hat{Q} = f \times \mathrm{id}_Q$:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times Q \arrow[l, "\mathrm{pr}_1"'] \arrow[d, "f\hat{Q}"] \arrow[dr, "\mathrm{pr}_2"] & \\ B & B \times Q \arrow[l, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"'] & Q \end{tikzcd}$$

As part of the theory of monoidal categories (Mac Lane, 1972, III.5, VII.1) there are canonical coherent isomorphisms $(A \times B) \times C \cong A \times (B \times C)$, $1 \times A \cong A \cong A \times 1$ which may be recast in the form

$$\begin{aligned} A\hat{B}\hat{C} &\cong A(B \times C)\hat{\ }, \\ 1\hat{A} &\cong A \cong A\hat{1}. \end{aligned} \tag{8}$$

Thus the representation $A \mapsto \hat{A}$ converts $\times$ into functorial composition at least up to isomorphism. We avoid a precise treatment of "up to isomorphism," which requires work (see Mac Lane, 1972), feeling that the theory to follow has been adequately motivated.

9. Definition. Given two functors $F, G: \mathscr{K} \to \mathscr{L}$, where $\mathscr{L}$ is a category with binary products, we define the functor $F \times G: \mathscr{K} \to \mathscr{L}$ by

$$\begin{aligned} A(F \times G) &= AF \times AG, \\ f(F \times G) &= fF \times fG. \end{aligned}$$

Motivated by the observation that led to (3) above, we now verify:

10. Proposition. *Let $\mathscr{K}$ be a category with binary products and a terminal object 1. Let Q, X_0, Y_0 be objects of $\mathscr{K}$. Then there exists a canonical injection from morphisms*

$$\lambda: Q \times X_0 \to Y_0$$

to natural transformations

$$\beta: \hat{Q}\hat{X}_0 \to \hat{Y}_0$$

given by

$$A\beta: A\hat{Q}\hat{X}_0 \cong A \times (Q \times X_0) \xrightarrow{\mathrm{id}_A \times \lambda} A\hat{Y}_0 . \tag{11}$$

Proof. To see that (11) describes a natural transformation, we must verify commutativity of the outer rectangle of

$$\begin{array}{ccccc}
A & & A\hat{Q}\hat{X}_0 \cong A \times (Q \times X_0) & \xrightarrow{\mathrm{id}_A \times \lambda} & A\hat{Y}_0 \\
{\scriptstyle f}\downarrow & & {\scriptstyle f\hat{Q}\hat{X}_0}\downarrow \qquad\qquad \downarrow{\scriptstyle f \times \mathrm{id}_{Q \times X_0}} & & \downarrow{\scriptstyle f \times \mathrm{id}_{Y_0}} \\
B & & B\hat{Q}\hat{X}_0 \cong B \times (Q \times X_0) & \xrightarrow[\mathrm{id}_B \times \lambda]{} & B\hat{Y}_0
\end{array}$$

But this is immediate since the canonical isomorphism $A\hat{Q}\hat{X}_0 \cong A \times (Q \times X_0)$ renders the left-hand square commutative.

Finally, λ is determined by its β since $\lambda = 1\beta$:

$$Q \times X_0 \cong 1\hat{Q}\hat{X}_0 \cong 1 \times (Q \times X_0) \xrightarrow{\mathrm{id} \times \lambda} 1\hat{Y}_0 \cong Y_0 . \quad \blacksquare$$

As a corollary of Theorem 12, which we establish in the next section, $\lambda \mapsto \beta$ is bijective when $\mathscr{K} = \mathbf{Set}$. However, for $\mathscr{K} = \mathbf{Vect}$, given $\lambda': Q \oplus X_0 \to X_0$, the transformation

$$\beta: \hat{Q}\hat{X}_0 \to \hat{Y}_0 \qquad \text{with} \quad A\beta(a, q, x) = (-a, \lambda(q, x))$$

is natural but is not induced by any λ in the fashion of (11).

With these preliminaries, we may now build on the motivation of (1)–(3) to give the promised definition of a process transformation. The passage from the map $\lambda: Q \times X_0 \to Y_0$ to a natural transformation will come to seem far less artificial when we turn to the serial composition of process transformations in Section 5.

12. Definition. Let A, B be objects of $\mathscr{K}$, and let X, Y be recursion processes in $\mathscr{K}$. A *restricted process transformation* $M: (A, X) \to (B, Y)$ in $\mathscr{K}$ is $M = (Q, \delta, \tau, \alpha, \beta)$, where

- (Q, δ) is an X-dynamics, the *state dynamics*,
- $\tau: A \to Q$ is the *initial state*,
- $\alpha: A \to B$ is the *initial throughput*,
- $\beta: \hat{Q}X \to Y$ is a natural transformation, the *output transformation.*

A *process transformation* $M: (A, X) \to (B, Y)$ in $\mathscr{K}$ is $M = (Q, \delta, \tau, \alpha, \beta)$ where (Q, δ) and τ are as above, but α, β are generalized to

$$\alpha: A \to BY@,$$
$$\beta: \hat{Q}X \to Y@.$$

A restricted process transformation induces a process transformation $M = (Q, \delta, \tau, \hat{\alpha}, \hat{\beta})$ be defining

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, "\hat{\alpha}"'] & B \arrow[d, "B\eta"] \\ & BY@ \end{tikzcd} \qquad \begin{tikzcd} \hat{Q}X \arrow[r, "\beta"] \arrow[dr, "\hat{\beta}"'] & Y \arrow[d, "\rho"] \\ & Y@ \end{tikzcd} \tag{13}$$

where ρ is the natural transformation defined by $AY \xrightarrow{A\eta Y} AY@Y \xrightarrow{A\mu_0} AY@$. In this sense, a restricted process transformation "is" a process transformation.

Recalling (1)–(3) we have:

14. Definition. Let $M = (Q, \delta, \tau, \alpha, \beta): (A, X) \to (B, Y)$ be a process transformation in $\mathscr{K}$. The *response* of M is the morphism $\gamma: AX@ \to BY@$ defined by the intertwined recursion

$$\begin{tikzcd} A \arrow[r, "A\eta^X"] \arrow[dr, "\alpha"'] & AX@ \arrow[d, "\gamma"] & & AX@X \arrow[ll, "A\mu_0{}^X"'] \arrow[d, "\binom{\gamma}{r}X"] \\ & BY@ & BY@Y@ \arrow[l, "B\mu^Y"] & (BY@ \times Q)X \arrow[l, "BY@\beta"] \end{tikzcd}$$

with r the reachability map $AX@ \to Q$ of (τ, δ).

Henceforth, we omit the superscripts on η, μ_0, and μ unless special emphasis seems needed.

For a restricted process transformation $M = (Q, \delta, \tau, \alpha, \beta)$ the response is defined to be that of the corresponding $\hat{M}$ and so, by (13), is given by diagram (15), on noting that

$$B\mu \cdot BY@\hat{\beta} = B\mu \cdot BY@\rho \cdot BY@\beta = B\mu_0 \cdot BY@\beta$$

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[d, "\alpha"'] & AX@ \arrow[d, "\gamma"] & & AX@X \arrow[ll, "A\mu_0"'] \arrow[d, "\binom{\gamma}{r}X"] \\ B \arrow[r, "B\eta"'] & BY@ & BY@Y \arrow[l, "B\mu_0"] & (BY@ \times Q)X \arrow[l, "BY@\beta"] \end{tikzcd} \tag{15}$$

16. Lemma. *Let $M: (A, X) \to (B, Y)$ be a process transformation $(Q, \delta, \tau, \alpha, \beta)$ and let $M': (A, X) \to (A, Y)$ be obtained from M by replacing α by $A\eta: A \to AY^@$. Let $\tilde{\alpha}$ be defined by*

$$\begin{tikzcd}
A \arrow[r, "A\eta"] \arrow[dr, "\alpha"'] & AY^@ \arrow[d, "\tilde{\alpha}"] & AY^@Y \arrow[l, "A\mu_0"'] \arrow[d, "\tilde{\alpha}Y"] \\
 & BY^@ & BY^@Y \arrow[l, "B\mu_0"']
\end{tikzcd}$$

Then the responses γ of M and γ' of M' are related by

$$\gamma = \tilde{\alpha} \cdot \gamma': AX^@ \to BY^@$$

so that we have

$$M = (A, X) \longrightarrow \boxed{\;M'\;} \xrightarrow{(A,Y)} \boxed{\;\alpha\;} \longrightarrow (B, Y)$$

has response $\gamma = \tilde{\alpha} \cdot \gamma'$.

Proof. On noting that $\tilde{\alpha}$ satisfies $\tilde{\alpha} \cdot A\mu = B\mu \cdot \tilde{\alpha}Y$, and that M and M' have the same reachability map r, we see that γ is defined as $\tilde{\alpha} \cdot \gamma'$ by the diagram:

$$\begin{tikzcd}
A \arrow[r, "A\eta^X"] \arrow[dr, "A\eta^Y"] \arrow[ddr, bend right, "\alpha"'] & AX^@ \arrow[d, "\gamma'"] & & AX^@X \arrow[ll, "A\mu_0"'] \arrow[d, "\binom{\gamma'}{\tau}X"] \\
 & AY^@ \arrow[d, "\tilde{\alpha}"] & AY^@Y \arrow[l, "A\mu"'] \arrow[d, "\tilde{\alpha}Y"'] & (AY^@ \times Q)X \arrow[l, "AY^@\beta"'] \arrow[d, "\tilde{\alpha}QX"] \\
 & BY^@ & BY^@Y \arrow[l, "B\mu"'] & (BY^@ \times Q)X. \arrow[l, "BY^@\beta"']
\end{tikzcd}$$

∎

In the classical study of monoids, any map $f: X_0 \to Y_0^*$ extends to a homomorphism $f^*: X_0^* \to Y_0^*$ by the inductive definition

$$\begin{aligned} f^*(\Lambda) &= \Lambda, \\ f^*(wx) &= f^*(w) \cdot f(x) \qquad \text{for} \quad w \in X^*, x \in X. \end{aligned}$$

This reveals f^* as the response of 1-state generalized sequential machine with

$\tau = \alpha = \mathrm{id}_1$,
$\delta: 1 \times X_0 \to 1$ which extends to the unique $r: 1 \times X^* \to 1$,
$\beta: 1 \times X_0 \to Y_0^* = f: X_0 \to Y_0^*$.

This motivates the following result, which (apart from the interpretation in terms of process transformations) is a version of a well-known construction concerning morphisms of algebraic theories (Manes, 1976):

17. Lemma. *Let $\mathscr{K}$ be a category with a terminal object* 1, *let X and Y be recursion processes in $\mathscr{K}$, and let $f: X \to Y^{@}$ be a natural transformation. We may then define a process transformation $(A, X) \to (A, Y)$ by*

$$\begin{aligned}
&\alpha = \mathrm{id}_A : A \to A,\\
&\tau: A \to 1,\\
&\delta: 1X \to 1 \qquad \textit{which extends to the unique } r: 1X^{@} \to 1,\\
&\beta: \hat{1}X \to Y^{@} = f: X \to Y^{@}.
\end{aligned}$$

the response $Af^{@}: AX^{@} \to AY^{@}$ of which is defined by the X-dynamorphic extension

$$\begin{tikzcd}
A \arrow[r, "A\eta^X"] \arrow[dr, "A\eta^Y"'] & AX^{@} \arrow[d, "Af^{@}"] & & AX^{@}X \arrow[ll, "A\mu_0^X"'] \arrow[d, "Af^{@}X"] \\
& AY^{@} & AY^{@}Y^{@} \arrow[l, "A\mu^Y"] & AY^{@}X \arrow[l, "AY^{@}f"]
\end{tikzcd}$$

Then $f^{@}: X^{@} \to Y^{@}$ is a natural transformation.

Proof. For $a: A \to B$, we must show that $Bf^{@} \cdot aX^{@} = aY^{@} \cdot Af^{@}$. We do this by observing from the following that both are induced as X-dynamorphisms by the same specifications.

$$\begin{tikzcd}
A \arrow[r, "A\eta^X"] \arrow[d, "a"] & AX^{@} \arrow[d, "aX^{@}"] & & AX^{@}X \arrow[ll, "B\mu_0^X"'] \arrow[d, "aX^{@}X"] \\
B \arrow[r, "B\eta^X"] \arrow[dr, "B\eta^Y"'] & BX^{@} \arrow[d, "Bf^{@}"] & & BX^{@}X \arrow[ll, "B\mu_0^X"'] \arrow[d, "Bf^{@}X"] \\
& BY^{@} & BY^{@}Y^{@} \arrow[l, "B\mu^Y"] & BY^{@}X \arrow[l, "BY^{@}f"]
\end{tikzcd} \tag{18}$$

commutes since η^X and $\mu_0{}^X$ are well known to be natural transformations. Again,

$$\begin{tikzcd}
A \arrow[r, "A\eta^X"] \arrow[dr, "A\eta^Y"'] \arrow[dd, "a"'] & AX^{@} \arrow[d, "Af^{@}"] & & AX^{@}X \arrow[ll, "A\mu_0^X"'] \arrow[d, "Af^{@}X"] \\
& AY^{@} \arrow[d, "aY^{@}"] & AY^{@}Y^{@} \arrow[l, "A\mu^Y"'] \arrow[d, "aY^{@}Y"] & AY^{@}X \arrow[l, "AY^{@}f"'] \arrow[d, "aY^{@}X"] \\
B \arrow[r, "B\eta^Y"] & BY^{@} & BY^{@}Y^{@} \arrow[l, "B\mu^Y"'] & BY^{@}X \arrow[l, "BY^{@}f"']
\end{tikzcd} \tag{19}$$

∎

20. COROLLARY. *The "memoryless code"* $a: A \to B$, $f: X \to Y^@$

$$(A, X) \longrightarrow \boxed{a/f} \longrightarrow (B, Y)$$

viewed as the process transformation (a/f): $(A, X) \to (B, Y)$: $(1,\ \delta: 1X \to 1,\ \tau: A \to 1,\ a: A \to B,\ f: X \to Y^@)$ *has response*

$$Bf^@ \cdot aX^@ = aY^@ \cdot Af^@. \quad \blacksquare$$

4. TREE TRANSFORMATIONS

In this section, we show that bottom-up tree transformations form a special case of process transformations, and then we provide a Yoneda-type lemma which provides further motivation for the introduction of the natural transformation $\beta: \hat{Q}X \to Y^@$.

1. DEFINITION. An *operator domain* Ω is a sequence $(\Omega_n \mid n \in \mathbf{N})$ of (possibly empty) disjoint sets. An *Ω-algebra* is a pair (Q, δ), where Q is a set and $\delta = (\delta_n)$ is a sequence of maps $\delta_n : Q^n \times \Omega_n \to Q$. We write δ_ω for $\delta(-, \omega): Q^n \to Q$ for $\omega \in \Omega_n$. Q is the *carrier* of the algebra.

Given Ω, we define a functor X_Ω : Set $\to$ Set by

$$QX_\Omega = \bigcup_{n \geqslant 0} Q^n \times \Omega_n, \tag{2}$$

while, for $Q \to Q'$,

$$hX_\Omega(q_1, \ldots, q_n, \omega) = (hq_1, \ldots, hq_n, \omega). \tag{3}$$

We now observe that an X_Ω-dynamics in the sense of 2.1 is just an Ω-algebra, and that an X_Ω-dynamorphism h: $(Q, \delta) \to (Q', \delta')$ is just an *Ω-homomorphism*, for the diagram in 2.1 unpacks to

$$h\delta_\omega(q_1, \ldots, q_n) = \delta'_\omega(hq_1, \ldots, hq_n) \qquad \text{for} \quad \omega \in \Omega_n,\ (q_1, \ldots, q_n) \in Q^n.$$

Moreover, X_Ω is a recursion process. $Ax_\Omega{}^@$ is the carrier of the well-known free Ω-algebra generated by A, and may be defined by the usual inductive definition (Birkhoff, 1935):

$$\begin{aligned} &A \subset AX_\Omega{}^@; \\ &\text{if } \omega \in \Omega_n,\ t_1, \ldots, t_n \in AX_\Omega{}^@, \text{ then } \omega t_1 \cdots t_n \in AX_\Omega{}^@; \\ &\text{nothing else is in } AX_\Omega{}^@. \end{aligned} \tag{4}$$

Thus the elements of $AX_\Omega{}^@$ may be regarded as finite rooted trees, with nodes of outdegree n labeled by elements of Ω_n, save that some leaves (nodes of outdegree 0) may be labeled by elements of A. We abbreviate $X_\Omega{}^@$ to T_Ω. We may define

$$\begin{aligned} A\eta&: A \to AT_\Omega, \qquad a \mapsto a, \\ A\mu_0&: AT_\Omega X_\Omega \to AT_\Omega: (t_1, \ldots, t_n, \omega) \mapsto \omega t_1 \cdots t_n. \end{aligned} \tag{5}$$

If (Q, δ) is any Ω-algebra and $\tau: A \to Q$ is any map

$$\begin{tikzcd} A \arrow[r, "A\eta"] \arrow[dr, "\tau"'] & AT_\Omega \arrow[d, "r"] & AT_\Omega X_\Omega \arrow[l, "A\mu_0"'] \arrow[d, "rX_\Omega"] \\ & Q & QX_\Omega \arrow[l, "\delta"] \end{tikzcd} \tag{6}$$

then the unique dynamorphic extension $r: AT_\Omega \to Q$ of τ is given by

$$\begin{aligned} r(a) &= \tau(a), \\ r(\omega t_1 \cdots t_n) &= \delta_\omega(rt_1, \ldots, rt_n). \end{aligned} \tag{7}$$

Note that this reduces to the dynamics $\delta: Q \times X_0 \to Q$ of 2.10 if we take $\Omega_1 = X_0$ while $\Omega_n = \varnothing$ for $n \neq 1$.

Suppose that Ω and Σ are two operator domains. We consider "bottom-up" (i.e., working from the leaves to the root) transformations of trees in AT_Ω into trees in BT_Σ:

8. Definition. Given operator domains Ω and Σ, and sets A and B, a *bottom-up tree transformation* $(A, \Omega) \to (B, \Sigma)$ is given by maps $\alpha: A \to B$, $\tau: A \to Q$, together with a sequence $\theta = (\theta_n)$ of maps

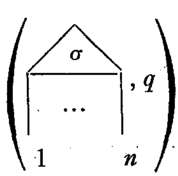
$$\theta_n : Q^n \times \Omega_n \to \{1, \ldots, n\}T_\Sigma \times Q. \tag{9}$$

The *response* of (α, τ, θ) is given by $\gamma: AT_\Omega \to BT_\Sigma \times Q$,

$$\gamma(a) = (\alpha(a), \tau(a)).$$

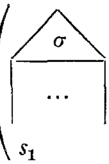
To define $\gamma(\omega t_1 \cdots t_n)$, let $\gamma(t_j) = (s_j, q_j)$. Then let

$$\theta(q_1, \ldots, q_n, \omega) =$$



so that

$$\gamma(\omega t_1 \cdots t_n) = \left(\quad , q \right).$$



Re-examining (9) we see that θ is defined by two families of maps,

$$\delta_n : Q^n \times \Omega_n \to Q \tag{10}$$

and

$$\beta_n : Q^n \times \Omega_n \to \bar{n}Y, \tag{11}$$

where $\bar{n}$ denotes $\{1,..., n\}$ and $Y = X_\Sigma$ is a functor Set $\to$ Set. The following result, in the style of the Yoneda Lemma (MacLane, 1971), provides justification for our formulation of β as a natural transformation.

12. THEOREM. *Let Ω be an operator domain, let Q be a set, and let Y be any functor* Set $\to$ Set. *Then there exists a canonical bijection*

$$\frac{\hat{Q}X_\Omega \xrightarrow{\beta} Y}{Q^n \times \Omega_n \xrightarrow{\beta_n} \bar{n}Y} \tag{13}$$

between natural transformation β and sequences (β_n) of functions. Mutually inverse passages are given by

$$\beta_n = Q^n \times \Omega_n \xrightarrow{k} \bar{n}\hat{Q}X_\Omega \xrightarrow{\bar{n}\beta} \bar{n}Y, \tag{14}$$

where $k(q_1 ,..., q_n , \omega) = ((1, q_1),..., (n, q_n), \omega)$,

$$A\beta\colon A\hat{Q}X_\Omega \to AY, \quad ((a_1 , q_1),..., (a_n , q_n), \omega) \mapsto (a_1 ,..., a_n)Y \cdot \beta_n(q_1 ,..., q_n , \omega) \tag{15}$$

(*where the n-tuple $(a_1 ,..., a_n) \in A^n$ is treated as a function $\bar{n} \to A$ so that the functorial action $(a_1 ,..., a_n)Y$ is a function $\bar{n}Y \to AY$*).

Proof. To see that (15) describes a natural transformation, we must verify

$$\begin{tikzcd}
(A \times Q)X_\Omega \arrow[r, "A\beta"] \arrow[d, "(h\times Q)X_\Omega"'] & AY \arrow[d, "hY"] \\
(B \times Q)X_\Omega \arrow[r, "B\beta"] & BY
\end{tikzcd}$$

for arbitrary $h\colon A \to B$. But starting from $(g, f, \omega) \in A^n \times Q^n \times \Omega_n$, the upper path yields $hY \cdot gY(\beta_n(f, \omega))$ and the lower path yields $(hg)Y(\beta_n(f, \omega))$, and these are equal since Y is a functor.

We now verify that (14) and (15) are inverse.
Now if $(\beta_n) \mapsto \beta \mapsto (\bar{\beta}_n)$, we have

$$\begin{aligned}\bar{\beta}_n(q_1, ..., q_n, \omega) &= \bar{n}\beta((1, q_1), ..., (n, q_n), \omega)\\ &= \bar{n}\beta(\mathrm{id}_{\bar{n}}, f, \omega) \qquad \text{for } \mathrm{id}_{\bar{n}} \in \bar{n}^{\bar{n}}, f = (q_1, ..., q_n) \in Q^n\\ &= \mathrm{id}_{\bar{n}} Y(\beta_n(f, \omega)) = \beta_n(q_1, ..., q_n, \omega).\end{aligned}$$

Conversely, if $\beta \mapsto \beta_n \mapsto \bar{\beta}$, then for $g \in A^n$ we have the naturality square

$$\begin{tikzcd}(\bar{n} \times Q)X_\Omega \arrow[r, "\bar{n}\beta"] \arrow[d, "(g \times Q)X_\Omega"'] & \bar{n}Y \arrow[d, "gY"] \\ (A \times Q)X_\Omega \arrow[r, "A\beta"] & AY\end{tikzcd}$$

so that

$$\begin{aligned}(A\bar{\beta})(g, f, \omega) &= (gY)(\beta_n(f, \omega))\\ &= (gY)(\bar{n}\beta)(\mathrm{id}_{\bar{n}}, f, \omega))\\ &= (A\beta)(g \times Q)\, X_\Omega(\mathrm{id}_{\bar{n}}, f, \omega)\\ &= (A\beta)(g, f, \omega). \quad \blacksquare\end{aligned}$$

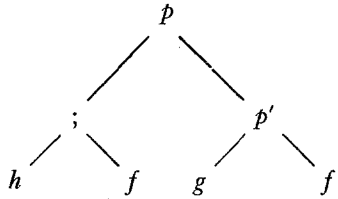
We thus conclude as follows:

16. Observation. A bottom-up tree transformation is simply a process transformation M: $(A, X) \to (B, Y)$ for $X = X_\Omega$, $Y = X_\Sigma$ for operator domains Ω and Σ.

17. Example. We now show how to capture what we believe to be the essential ideas of Reynolds' (1977) "Semantics of the Domain of Flow Diagrams" by giving a succinct account of the relation between general flow diagrams and linear flow diagrams which, we believe, provides the paradigm for the other relations discussed in that paper. However, it should be noted that Reynolds does not, as we do here, work in the category of sets, and interesting technical questions may remain to be resolved in extending our treatment to other categories in which morphisms are constrained to preserve appropriate ordering relations. We fix a set P of predicate symbols and a set F of function symbols. A general flow diagram may be represented by a Σ-tree where

$$\Sigma_0 = F, \qquad \Sigma_1 = \varnothing, \qquad \Sigma_2 = P \cup \{\, ; \,\} \tag{18}$$

and we interpret the following element of $\varnothing T_\Sigma$



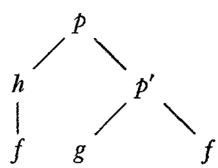
(19)

as "If the p-test yields true, execute h then f; whereas if the test yields false, carry out the p'-test, executing g if the outcome is true, f if the outcome is false."

A linear flow diagram is one in which we cannot compose arbitrary operations using " ; ", but instead apply one f at a time. They correspond to Ω-trees where

$$\Omega_0 = F, \qquad \Omega_1 = F, \qquad \Omega_2 = P, \tag{20}$$

and (19) corresponds to the following element of $\varnothing T_\Omega$



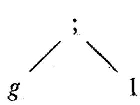
(21)

We now show that that transformation from linear flow diagrams (as represented by Ω-tree) to general flow diagrams (as represented by Σ-trees) is given by a *pure* (i.e., Q has only one element) tree transformation, i.e., (recalling (9)) by a sequence of maps

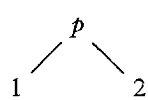
$$\theta_n : \Omega_n \to \{1,..., n\}\, T_\Sigma$$

which in this case take the form

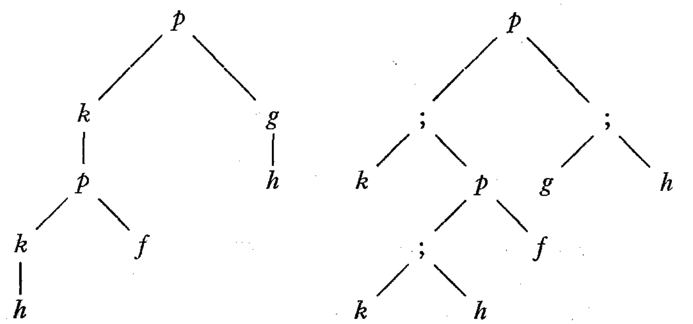
$$\theta_0(f) = f$$

$\theta_1(g) =$ 

(22)

$\theta_2(p) =$ 

The response $\varnothing T_\Omega \to \varnothing T_\Sigma$ does indeed transform (21) into (19), and the reader may see that it also yields the following typical transformation:



(23)

Now Reynolds provides for each direct (respectively, continuation) semantics for general flow diagrams a corresponding semantics for linear flow diagrams. But each semantics for a general (respectively, linear) flow diagram is nothing more nor less than a Σ- (respectively, Ω-) algebra. Any particular choice of a transformation of semantics which "preserves meaning" with respect to a particular transformation of flow diagrams is subsumed in the following result (which works just as well when T_Σ and T_Ω are replaced by arbitrary algebraic theories T_1 and T_2):

24. Proposition. *Let Ω and Σ be operator domains, and let $\xi: RX_\Sigma \to R$ be a given Σ-algebra. Further, let the family of maps*

$$\theta_n : \Omega_n \to \{1,\ldots, n\}\, T_\Sigma$$

define a pure tree transformation. Then there exists an Ω-algebra $\delta: RX_\Omega \to Q$ such that the result of running δ on any Ω-tree equals the result of running ξ on the transformed Σ-tree.

Proof. By (13), for the case $Q = \{1\}$, θ_n is equivalent to a natural transformation

$$\theta: X_\Omega \to T_\Sigma$$

yielding, in particular, the map

$$R\theta: RX_\Omega \to RT_\Sigma . \tag{25}$$

Now we define the run map $\xi^@: RT_\Sigma \to R$ of (R, ξ) by the diagram (compare (6))

$$\begin{tikzcd} R \arrow[r, "R\eta^\Sigma"] \arrow[dr, "\mathrm{id}_R"'] & RT_\Sigma \arrow[d, "\xi^@"] & RT_\Sigma X_\Sigma \arrow[l, "R\mu_0"'] \arrow[d, "\xi^@ X_\Sigma"] \\ & R & RX_\Sigma \arrow[l, "\xi"'] \end{tikzcd} \tag{26}$$

and we may then define an Ω-algebra (R, δ) by

$$\delta = RX_\Omega \xrightarrow{R\theta} RT_\Sigma \xrightarrow{\xi@} R. \tag{27}$$

To show that δ has the claimed property, we must look at the response $\gamma\colon RT_\Omega \to RT_\Sigma$ of the process transformation with $A = B = R$ and $\alpha = \mathrm{id}_R$, $Q = 1$, and $\tau\colon R \to 1$, and with $X = X_\Omega$, $Y = X_\Sigma$, and $\beta = \theta\colon X \to Y@$. We have

$$\begin{tikzcd}
R \arrow[r, "R\eta^\Omega"] \arrow[dr, "R\eta^\Sigma"'] & RT_\Omega \arrow[d, "\gamma"] & & RT_\Omega X_\Omega \arrow[ll, "R\mu_0{}^\Omega"'] \arrow[d, "\gamma X_\Omega"] \\
& RT_\Sigma & RT_\Sigma T_\Sigma \arrow[l, "R\mu^\Sigma"'] & RT_\Sigma X_\Omega \arrow[l, "RT_\Sigma\theta"']
\end{tikzcd} \tag{28}$$

We have to show that $\delta@ = RT_\Omega \xrightarrow{\gamma} RT_\Sigma \xrightarrow{\xi@} R$ to complete the proof of the proposition. But this is immediate from the following diagram:

$$\begin{tikzcd}
R \arrow[r, "R\eta^\Omega"] \arrow[dr, "R\eta^\Sigma"'] \arrow[ddr, bend right, "\mathrm{id}_R"'] & RT_\Omega \arrow[d, "\gamma"] \arrow[dr, phantom, "\mathrm{II}"] & & RT_\Sigma X_\Sigma \arrow[ll, "R\mu_O"'] \arrow[d, "\gamma X_\Omega"] \\
& RT_\Sigma \arrow[d, "\xi^@"] \arrow[dr, phantom, "\mathrm{IV}"] & RT_\Sigma T_\Sigma \arrow[l, "R\mu"'] \arrow[d, "\xi^@ T_\Sigma"] \arrow[dr, phantom, "\mathrm{V}"] & RT_\Sigma X_\Omega \arrow[l, "RT_\Sigma\theta"'] \arrow[d, "\xi^@ X_\Omega"] \\
& R & RT_\Sigma \arrow[l, "\xi^@"'] & RX_\Omega \arrow[l, "R\theta"'] \arrow[ll, bend left, "\delta"]
\end{tikzcd} \tag{29}$$

where I and II are just (28), III and IV extend (26), V is a naturality square for θ, and VI is the definition of δ. Thus $\xi@ \cdot \gamma$ satisfies the diagram which defines $\delta@$ uniquely. ∎

Since it is an immediate generalization of the above, we may state the following without further proof:

30. THEOREM. *Let $M = (1, \delta, \tau, \mathrm{id}_A, \beta)\colon (A, X) \to (A, Y)$ be a pure process transformation ($Q = 1$) with response $\gamma\colon AX@ \to AY@$, and let (A, ξ) be a Y-dynamics. Then the X-dynamics (A, δ) defined by*

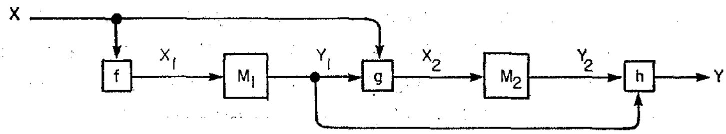
$$\delta = AX \xrightarrow{A\beta} AY@ \xrightarrow{\xi@} A$$

satisfies the equation

$$\delta@ = AX@ \xrightarrow{\gamma} AY@ \xrightarrow{\xi@} A. \quad ∎$$

5. Behavior of Loop-Free Networks

Our development in this section is motivated by the study of the *cascade connection* of sequential machines as shown in (1) (Arbib, 1968).



(1)

In this motivating example, we assume a single initial state, so that α may be omitted. Here, then, $M_i = (Q_i, \delta_i, \tau_i, \lambda_i)\colon X_i \to Y_i$ are Mealy machines, and f, g, h are auxiliary functions of the form

$$\begin{aligned} f&\colon X \to X_1, \\ g&\colon X \times Y_1 \to X_2, \\ h&\colon Y_1 \times Y_2 \to Y. \end{aligned} \tag{2}$$

The formal definition of the cascade connection of M_1 and M_2 via (f, g, h) is then the Mealy machine $M = (Q, \delta, \tau, \lambda)\colon X \to Y$ defined by

$$\begin{aligned} Q &= Q_1 \times Q_2, \\ \delta(q_1, q_2, x) &= (\delta_1(q_1, fx), \delta_2(q_2, g(x, \lambda_1(q_1, fx)))), \\ \tau &= (\tau_1, \tau_2), \\ \lambda(q_1, q_2, x) &= h(\lambda_1(q_1, fx), \lambda_2(q_2, g(x, \lambda_1(q_1, fx)))). \end{aligned} \tag{3}$$

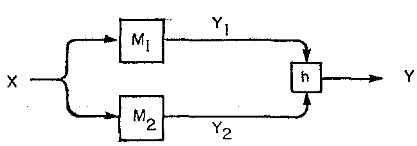
As can readily be seen the *serial connection* (4) and *parallel connection* (5) may be obtained as special cases.

$$X_1 \longrightarrow \boxed{M_1} \xrightarrow{Y_1 = X_2} \boxed{M_2} \longrightarrow Y_2 \tag{4}$$

which is obtained from (1) on taking

$$X = X_1, \qquad Y_1 = X_2, \qquad Y_2 = Y,$$

$$f = \mathrm{id}_{X_1}; \qquad g = \mathrm{pr}_2, \quad (x, y) \mapsto y; \qquad h = \mathrm{pr}_2, \quad (y_1, y_2) \mapsto y_2,$$



(5)

which is obtained from (1) on taking

$$X = X_1 = X_2,$$

$$f = \mathrm{id}_{X_1}; \qquad g = \mathrm{pr}_1, \quad (x, y) \mapsto y; \qquad \text{arbitrary } h.$$

It is also well known that the behavior of an arbitrary cascade connection can be reconstructed by a loop-free network built up using only series and parallel connections. We provide an analogous result in a more general setting. We work, for simplicity, with *restricted* process transformations.

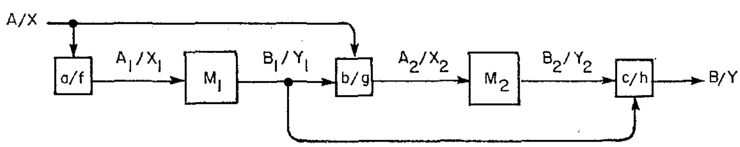
6. Definition. Let $M_1 = (Q_1, \delta_1, \tau_1, \alpha_1, \beta_1)\colon (A_1, X_1) \to (B_1, Y_1)$ and $M_2 = (Q_2, \delta_2, \tau_2, \alpha_2, \beta_2)\colon (A_2, X_2) \to (B_2, Y_2)$ be restricted process transformations in $\mathscr{K}$. Let f, g, h be natural transformations

$$f\colon X \to X_1, \qquad g\colon X \times Y_1 \to X_2, \qquad h\colon Y_1 \times Y_2 \to Y, \tag{7}$$

where X, Y are also recursion processes; and let a, b, c be morphisms

$$a\colon A \to A_1, \qquad b\colon A \times B_1 \to A_2, \qquad c\colon B_1 \times B_2 \to B. \tag{8}$$

Then the *cascade connection* of M_1 and M_2 with respect to (f, g, h) and (a, b, c) is the restricted process transformation $M = (Q, \delta, \tau, \alpha, \beta)\colon (A, X) \to (B, Y)$ represented in the block diagram



(9)

and is defined as follows[5]:

$$Q = Q_1 \times Q_2$$

$$\begin{tikzcd}
A_1 \arrow[rr, "\tau_1"] & & Q_1 \\
A \arrow[u, "a"] \arrow[rr, dashed, "\tau"] \arrow[d, "\binom{\mathrm{id}_A}{\alpha_1 a}"'] & & Q_1 \times Q_2 \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\
A \times B_1 \arrow[r, "b"] & A_2 \arrow[r, "\tau_2"] & Q_2
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[rr, "\alpha_1"] & & B_1 & \\
A \arrow[u, "a"] \arrow[rr, dashed] \arrow[d, "\binom{\mathrm{id}_A}{\alpha_1 a}"'] \arrow[rrr, bend right, "\alpha"'] & & B_1 \times B_2 \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \arrow[r, "c"] & B \\
A \times B_1 \arrow[r, "b"] & A_2 \arrow[r, "\alpha_2"] & B_2 &
\end{tikzcd} \tag{10}$$

$$\{\tau(s) = (\tau_1 a(s), \tau_2 b(s, \alpha_1 a(s)))\} \qquad \{\alpha(s) = c(\alpha_1 a(s), \alpha_2 b(s, \alpha_1 a(s)))\}$$

[5] To aid comprehension we place the classical formula in parentheses below each diagram.

If we now define

$$\Gamma = \hat{Q}X \xrightarrow{\hat{Q}f} \hat{Q}X_1 \cong \hat{Q}_2\hat{Q}_1X_1 \xrightarrow{\hat{Q}_2\beta_1} \hat{Q}_2Y_1\,,$$

$$\Delta = \hat{Q}X \xrightarrow{\binom{\hat{\mathrm{pr}}_2X}{\Gamma}} \hat{Q}_2X \times \hat{Q}_2Y_1 \xrightarrow{\hat{Q}_2g} \hat{Q}_2X_2\,, \tag{11}$$

$$\{\Gamma(q_1, q_2, x) = (q_2, \beta_1(q_1, f(x))); \ \Delta(q_1, q_2, x) = (q_2, g(x, \beta_1(q_1, f(x))))\}$$

then δ and β are defined by

$$\begin{tikzcd}
Q_1X \arrow[r, "Q_1f"] & Q_1X_1 \arrow[r, "\delta_1"] & Q_1 \\
QX \arrow[u, "\mathrm{pr}_1X"] \arrow[rr, dashed, "\delta"] \arrow[d, "1\Delta"'] & & Q \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\
Q_2X_2 \arrow[rr, "\delta_2"] & & Q_2
\end{tikzcd} \tag{12}$$

$$\{\delta(q_1, q_2, x) = (\delta_1(q_1, f(x)), \delta_2(q_2, g(x, \beta_1(q_1, f(x)))))\},$$

$$\begin{tikzcd}
\hat{Q}_1X \arrow[r, "\hat{Q}_1f"] & \hat{Q}_1X_1 \arrow[r, "\beta_1"] & Y_1 & \\
QX \arrow[u, "\hat{\mathrm{pr}}_1X"] \arrow[rr, dashed] \arrow[rrr, bend left, "\beta"] \arrow[d, "\Delta"'] & & Y_1 \times Y_2 \arrow[u, "\mathrm{pr}_1"'] \arrow[r, "h"'] \arrow[d, "\mathrm{pr}_2"] & Y \\
\hat{Q}_2X_2 \arrow[rr, "\beta_2"'] & & Y_2 &
\end{tikzcd} \tag{13}$$

$$\{\beta(q_1, q_2, x) = h(\beta_1(q_1, f(x)), \beta_2(q_2, g(x, \beta_1(q_1, f(x)))))\}.$$

Following the example of (4) and (5), we may read off the following definitions of the serial and parallel connections of two process transformations.

14. Definition. Given restricted process transformations $M_1 : (A, X) \to (B, Y)$ and $M_2 : (B, Y) \to (C, Z)$, their *serial connection* $M_2M_1 : (A, X) \to (C, Z)$ is represented by the block diagram

$$(A, X) \longrightarrow \boxed{\;M_1\;} \xrightarrow{Y} \boxed{\;M_2\;} \longrightarrow (C, Z) \tag{15}$$

and is the cascade connection with auxiliaries

$$f = \mathrm{id}_X : X \to X; \qquad g = \mathrm{pr}_2 : X \times Y \to Y; \qquad h = \mathrm{pr}_2 : Y \times Z \to Z,$$

$$a = \mathrm{id}_A : A \to A; \qquad b = \mathrm{pr}_2 : A \times B \to B; \qquad c = \mathrm{pr}_2 : B \times C \to C.$$

Thus $M_2M_1 = (Q, \delta, \tau, \alpha, \beta)$, where

$$
\begin{aligned}
&Q = Q_1 \times Q_2\,,\\
&\tau = \begin{pmatrix}\tau_1\\ \tau_2\alpha_1\end{pmatrix}: A \to Q_1 \times Q_2\,,\\
&\{\tau(s) = (\tau_1(s), \tau_2\alpha_1(s))\},\\
&\alpha = A \xrightarrow{\alpha_1} B \xrightarrow{\alpha_2} C,\\
&\{\alpha(s) = \alpha_2\alpha_1(s)\},
\end{aligned} \tag{16}
$$

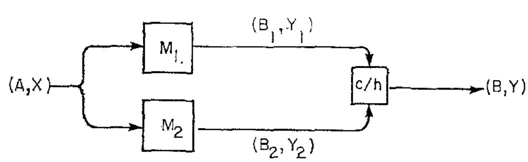
$$
\begin{tikzcd}
Q_1X \arrow[rr, "\delta_1"] & & Q_1 \\
QX \arrow[u, "\mathrm{pr}_1X"] \arrow[rr, "\delta"] \arrow[d, "\cong"'] & & Q \arrow[u, "\mathrm{pr}_1"'] \arrow[dd, "\mathrm{pr}_2"] \\
Q_2Q_1X \arrow[d, "Q_2\beta_1"'] & & \\
Q_2Y \arrow[rr, "\delta_2"] & & Q_2
\end{tikzcd}
$$

$$\{\delta(q_1, q_2, x) = (\delta_1(q_1, x), \delta_2(q_2, \beta_1(q_1, x)))\},$$

$$\beta = \hat{Q}X \cong \hat{Q}_2\hat{Q}_1X \xrightarrow{\hat{Q}_2\beta_1} \hat{Q}_2Y \xrightarrow{\beta_2} Z,$$

$$\{\beta(q_1, q_2, x) = \beta_2(q_2, \beta_1(q_1, x))\}.$$

17. Definition. Given restricted process transformations $M_i : (A, X) \to (B_i, Y_i)$ $(i = 1, 2)$, a recursion process Y, a natural transformation h: $Y_1 \times Y_2 \to Y$, and a morphism c: $B_1 \times B_2 \to Y$, the *(c/h)-parallel connection of M_1 and M_2* is M: $(A, X) \to (B, Y)$ represented by the block diagram



(18)

and is the cascade connection with auxiliaries

$$
\begin{array}{lll}
f = \mathrm{id}: X \to X; & g = \mathrm{pr}_1 : X \times Y_1 \to X; & h: Y_1 \times Y_2 \to Y,\\
a = \mathrm{id}: A \to A; & b = \mathrm{pr}_1 : A \times B_1 \to A; & c: B_1 \times B_2 \to B.
\end{array}
$$

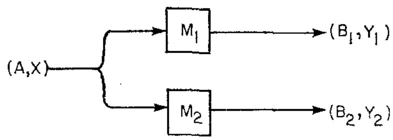
Thus $M_1 \times M_2 = (Q, \delta, \tau, \alpha, \beta)$, where

$$\begin{aligned} Q &= Q_1 \times Q_2, \\ \tau &= \begin{pmatrix}\tau_1 \\ \tau_2\end{pmatrix}: A \to Q_1 \times Q_2, \qquad\qquad (19) \\ \alpha &= c \cdot \begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}: A \to B_1 \times B_2 \to B, \end{aligned}$$

$$\begin{tikzcd} QX \arrow[r, "\delta"] \arrow[d, "\mathrm{pr}_i X"'] & Q \arrow[d, "\mathrm{pr}_i"] \\ Q_i X \arrow[r, "\delta_i"] & Q_i \end{tikzcd} \qquad \begin{tikzcd} \hat{Q}X \arrow[r] \arrow[rr, bend left, "\beta"] \arrow[d, "\hat{\mathrm{pr}}_i X"'] & Y_1 \times Y_2 \arrow[r, "h"'] \arrow[d, "\mathrm{pr}_i"] & Y \\ \hat{Q}_i X \arrow[r, "\delta_i"] & Y_i & \end{tikzcd}$$

$$\{\delta(q_1, q_2, x) = (\delta_1(q_1, x), \delta_2(q_2, x)); \beta(q_1, q_2, x) = h(\beta_1(q_1, x), \beta_2(q_2, x))\}.$$

It would be pleasant to replace (18) by the parallel connection represented by



(20)

However, this requires $Y_1 \times Y_2$—rather than just Y_1 and Y_2 separately—to be a recursion process. This can fail even in Set (see Adámek and Trnková, 1978, p. 8) which is why we prefer (18) to (20), but the following example provides some positive impetus for (20)!

21. EXAMPLE. Let Ω and Σ be operator domains, and let X_Ω and X_Σ be the corresponding recursion processes Set $\to$ Set. Then

$$\begin{aligned} Q(X_\Omega \times X_\Sigma) &= QX_\Omega \times QX_\Sigma \\ &= \coprod_{m \geqslant 0} Q^m \times \Omega_m \times \coprod_{m \geqslant 0} Q^n \times \Sigma_n \\ &\cong \coprod_{k \geqslant 0} Q^k \times \left(\coprod_{m+n=k} \Omega_m \times \Sigma_n\right). \end{aligned}$$

Thus $X_\Omega \times X_\Sigma$ is a recursion process in this case, and is of the form X_Ψ, where the operator domain Ψ is the *convolution* $\Omega * \Sigma$ of Ω and Σ defined by

$$(\Omega * \Sigma)_k = \{(\omega, \sigma) \mid \omega \in \Omega_m, \sigma \in \Sigma_n \text{ with } m + n = k\},$$

We devote the rest of this section to studying the behavior of these various connections.

22. Definition. The *behavior* of a restricted process transformation M is the quadruple $(r, \alpha, \beta, \gamma)$ comprising

$r\colon AX^{@} \to Q,$	the reachability map,
$\alpha\colon A \to B,$	the initial throughput,
$\beta\colon \hat{Q}X \to Y,$	the output transformation,
$\gamma\colon AX^{@} \to BY^{@},$	the response.

23. Theorem. *Given $M_1 : (A, X) \to (B, Y)$ and $M_2 : (B, Y) \to (C, Z)$ with behaviors $(r_1, \alpha_1, \beta_1, \gamma_1)$ and $(r_2, \alpha_2, \beta_2, \gamma_2)$, respectively, then the behavior $(r, \alpha, \beta, \gamma)$ of their serial connection $M_2M_1 : (A, X) \to (C, Z)$ is given by*

$$\begin{aligned} r &= \binom{r_1}{r_2\gamma_1}\colon AX^{@} \to Q_1 \times Q_2, \\ \alpha &= \alpha_2\alpha_1\colon A \to C, \\ \beta &= \beta_2 \cdot \hat{Q}_2\beta_1, \\ \gamma &= \gamma_2\gamma_1 . \end{aligned} \tag{24}$$

Proof. The expressions for α and β are immediate from Definition 14. We first recall the diagram defining γ_1:

$$\begin{tikzcd}
A \arrow[r, "A\eta"] \arrow[d, "\alpha"'] & AX^{@} \arrow[d, "\gamma_1"] & & AX^{@}X \arrow[ll, "A\mu_0"'] \arrow[d, "\binom{\gamma_1}{r_1}X"] \\
B \arrow[r, "B\eta"'] & BY^{@} & BY^{@}Y \arrow[l, "B\mu_0"] & (BY^{@} \times Q_1)X \arrow[l, "BY^{@}\beta_1"]
\end{tikzcd} \tag{25}$$

and that defining r:

$$\begin{tikzcd}
A \arrow[r, "A\eta"] \arrow[dr, "\binom{\tau_1}{\tau_2\alpha_1}"'] & AX^{@} \arrow[d, "r"] & AX^{@}X \arrow[l, "A\mu_0"'] \arrow[d, "rX"] \\
& Q_1 \times Q_2 & (Q_1 \times Q_2)X \arrow[l, "\delta"']
\end{tikzcd} \tag{26}$$

To see that $\mathrm{pr}_1 \cdot r = r_1$, we simply inspect the diagram

$$\begin{tikzcd}
A \arrow[r, "A\eta"] \arrow[dr, "\binom{\tau_1}{\tau_2\alpha_1}"'] \arrow[ddr, bend right, "\tau_1"'] & AX^{@} \arrow[d, "r"] & AX^{@}X \arrow[l, "A\mu_0"'] \arrow[d, "rX"] \\
& Q \arrow[d, "\mathrm{pr}_1"] & QX \arrow[l, "\delta"'] \arrow[d, "\mathrm{pr}_1X"] \\
& Q_1 & Q_1X \arrow[l, "\delta_1"']
\end{tikzcd} \tag{27}$$

To prove $\mathrm{pr}_2 \cdot r = r_2 \cdot \gamma_1$, we show that each is defined by intertwined recursion on the same specifications:

$$
\begin{tikzcd}
A \arrow[r, "A\eta"] \arrow[d, "\alpha_1"'] & AX^{@} \arrow[d, "\gamma_1"] & & AX^{@}X \arrow[ll, "A\mu_0^X"'] \arrow[d, "\left(\begin{smallmatrix}\gamma_1\\ r_1\end{smallmatrix}\right)X"] \arrow[ddd, bend left=60, "\left(\begin{smallmatrix}r_2\gamma_1\\ r_1\end{smallmatrix}\right)X"] \\
B \arrow[r, "B\eta"] \arrow[ddr, "\tau_2"'] \arrow[dr, phantom, "\mathrm{I}"] & BY^{@} \arrow[dd, "r_2"] \arrow[r, phantom, "\mathrm{II}" near start, shift right=4ex] & BY^{@}Y \arrow[l, "B\mu_0^Y"'] \arrow[d, "r_2Y"] \arrow[r, phantom, "\mathrm{III}", shift right=4ex] & (BY^{@}\times Q_1)X \arrow[l, "BY^{@}\beta_1"'] \arrow[dd, "r_2\hat{Q}_1X"] \\
 & & Q_2Y \arrow[dl, "\delta_2"'] \arrow[d, phantom, "\mathrm{IV}"] & \\
 & Q_2 & Q \arrow[l, "pr_2"] & Q_2\hat{Q}_1X \arrow[l, "\delta"] \arrow[ul, "Q_2\beta_1"']
\end{tikzcd}
\tag{28}
$$

where the upper rectangles commute by the definition of γ_1, I and II commute by the definition of r_2, III commutes by the naturality of β_1, and IV commutes by the definition of δ.

$$
\begin{tikzcd}
A \arrow[r, "A\eta"] \arrow[d, "\alpha_1"'] \arrow[dr, "\tau"] & AX^{@} \arrow[d, "r"] & AX^{@}X \arrow[l, "A\mu_0^X"'] \arrow[d, "rX"] \arrow[dd, bend left=60, "\left(\begin{smallmatrix}pr_2\cdot r\\ r_1\end{smallmatrix}\right)X"] \\
B \arrow[dr, "\tau_2"'] & Q \arrow[d, "pr_2"] & QX \arrow[l, "\delta"'] \arrow[d, "\left(\begin{smallmatrix}pr_2\\ pr_1\end{smallmatrix}\right)X"] \\
 & Q_2 & Q_2\hat{Q}_1X \arrow[l, "\delta_2\cdot Q_2\beta_1"']
\end{tikzcd}
\tag{29}
$$

Comparing (28) and (29), we see that $\mathrm{pr}_2 \cdot r = r_2 \cdot \gamma_1$. To show that $\gamma = \gamma_2\gamma_1$, we must verify that

$$
\begin{tikzcd}
A \arrow[r, "A\eta^X"] \arrow[d, "\alpha"'] & AX@ \arrow[d, "\gamma_2\gamma_1"] & & AX@X \arrow[ll, "A\mu_0^X"'] \arrow[d, "\left(\begin{smallmatrix}\gamma_2\gamma_1\\ r\end{smallmatrix}\right)X"] \\
C \arrow[r, "C\eta^Z"] & CZ@ & CZ@Z \arrow[l, "C\mu_0^Z"'] & (CZ@ \times Q)X \arrow[l, "cz@\beta"']
\end{tikzcd}
\tag{30}
$$

which is accomplished in the following diagram, which makes use of our verification that $r = \left(\begin{smallmatrix}r_1\\ r_2\gamma_1\end{smallmatrix}\right)$.

$$
\begin{tikzcd}
A \arrow[rr, "A\eta^X"] \arrow[dr, "\alpha_1"] \arrow[ddd, "\alpha"'] \arrow[rr, phantom, "\mathrm{II}", shift right=5ex] & & AX@ \arrow[d, "\gamma_1"] \arrow[rr, phantom, "\mathrm{III}", shift right=5ex] & & AX@X \arrow[ll, "A\mu_0"'] \arrow[d, "\left(\begin{smallmatrix}\gamma_1\\ r_1\end{smallmatrix}\right)X"] \\
 & B \arrow[r, "B\eta^X"'] \arrow[ddl, "\alpha_2"] \arrow[l, phantom, "\mathrm{I}", shift right=8ex] \arrow[d, phantom, "\mathrm{IV}" near end, shift left=4ex] & BY@ \arrow[dd, "\gamma_2"] \arrow[d, phantom, "\mathrm{V}" near end, shift left=4ex] & BY@Y \arrow[l, "B\mu_0^Y"] \arrow[d, "\left(\begin{smallmatrix}\gamma_2\\ r_2\end{smallmatrix}\right)Y"] & (BY@ \times Q_1)X \arrow[l, "BY@\beta_1"'] \arrow[dd, "\left(\begin{smallmatrix}\gamma_2\\ r_2\end{smallmatrix}\right)\hat{Q}_1X"] \arrow[dl, phantom, "\mathrm{VI}"] \\
 & & & (CZ@ \times Q_2)Y \arrow[d, "cz@\beta_2"'] \arrow[dr, phantom, "\mathrm{VII}"] & \\
C \arrow[rr, "C\eta^Z"'] & & CZ@ & CZ@Z \arrow[l, "C\mu_0^Z"] & CZ@\hat{Q}_2\hat{Q}_1X \arrow[l, "cz@\beta"] \arrow[ul, "cz@\hat{Q}_2\beta_1"']
\end{tikzcd}
\tag{31}
$$

In (31), I is the definition of α, II and III define γ_1, IV and V define γ_2, VI commutes by the naturality of β_1, and VII defines $CZ^{@}\beta$. Comparing (30) and (31), we conclude that $\gamma = \gamma_2\gamma_1$. ∎

We state the next result without proof, since the proof is akin to, but simpler than, the proof we have just given for the serial composition.

32. THEOREM. *Given $M_1 : (A, X) \to (B_1, Y_1)$ and $M_2 : (A, X) \to (B_2, Y_2)$ with behaviors $(r_1, \alpha_1, \beta_1, \gamma_1)$ and $(r_2, \alpha_2, \beta_2, \gamma_2)$, respectively, then the behavior of their (c/h)-parallel connection M: $(A, X) \to (B, Y)$ is given by*

$$\begin{aligned} r &= \binom{r_1}{r_2}, \\ \alpha &= \binom{\alpha_1}{\alpha_2}, \\ \beta &= h \begin{pmatrix} \beta_1\, \mathrm{pr}_1 X \\ \beta_2\, \mathrm{pr}_2 X \end{pmatrix}, \end{aligned} \tag{33}$$

while the response γ is uniquely determined by

$$\begin{tikzcd} A \arrow[r, "A\eta^X"] \arrow[d, "\binom{\alpha_1}{\alpha_2}"'] & AX^{@} \arrow[dd, "\gamma"] & & AX^{@}X \arrow[ll, "A\mu_0^X"'] \arrow[dd, "\begin{pmatrix} BY^{@}\beta_1\cdot\binom{\gamma}{r_1}X \\ BY^{@}\beta_2\cdot\binom{\gamma}{r_2}X \end{pmatrix}"] \\ B_1 \times B_2 \arrow[d, "c"'] & & & \\ B \arrow[r, "B\eta^Y"] & BY^{@} & BY^{@}Y \arrow[l, "B\mu_0^Y"'] & BY^{@}Y_1 \times BY^{@}Y_2 \arrow[l, "BY^{@}h"'] \end{tikzcd} \tag{34}$$
∎

Diagram (34) corresponds to the recursion

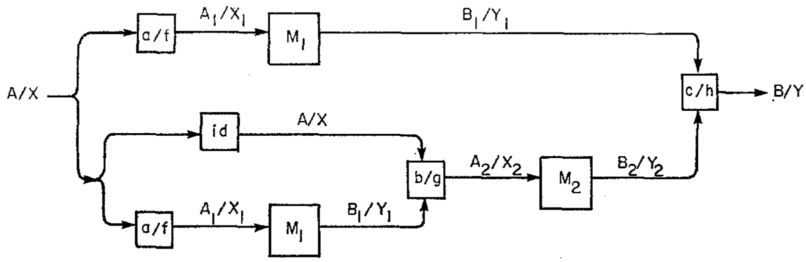
$$\begin{aligned} \gamma(\Lambda) &= \Lambda, \\ \gamma(wx) &= \gamma(w) \cdot h(\beta_1(r_1(w), x), \beta_2(r_2(w), x)). \end{aligned}$$

In the classical case, we can extend h to h^*: $(Y_1 \times Y_2)^* \to Z$, where $(Y_1 \times Y_2)^* \cong \coprod_{n\geqslant 0} Y_1^n \times Y_2^n$, so that we also have the formula

$$\gamma(w) = h^*(\gamma_1(w), \gamma_2(w)).$$

However, as Example 21 emphasizes, no similarly convenient extension of h is known to be available in the general case.

We close this section by noting that the cascade connection (9) can be *simulated* by the loop-free network shown below in (35). By this, we mean that (35) has the same response γ as that of (9)—although we do not burden the reader with the diagram-chasing involved in the proof.



(35)

We start by forming two copies of $M_1 \cdot (a/f)$, the series connection of M_1 with the memoryless code (a/f) of Corollary 3.20. Then $M_1 \cdot (a/f)$ has response $\gamma_1 \cdot A_1 f^{@} \cdot aX^{@}$. Then, as discussed in 3.17, we may regard the identity natural transformation id: $X \to X$ as a process transformation, and its response is the identity natural transformation $X^{@} \to X^{@}$. We then form the (b/q)-parallel connection of id and $M_1 \cdot (a/f)$—call the result M_3. Finally, we form the series connection $M_2 \cdot M_3$, and then the (c/h)-parallel connection of $M_1 \cdot (a/f)$ and $M_2 \cdot M_3$.

6. Linear Systems

There are two formalizations of linear systems in the recursion process literature. The *decomposable system approach* (Arbib and Manes, 1974b) takes $X = \mathrm{id}_{\mathrm{Vect}}$, and represents a linear system with input space A, state-space Q, and output space B, and with input map $G: A \to Q$, dynamics $F: Q \to Q$, and output map $H: Q \to B$ as

$$\begin{aligned} \tau &= G: A \to Q, \\ \delta &= F: QX = Q \to Q, \\ \beta &= H: Q \to B. \end{aligned} \tag{1}$$

The *coproduct approach* (Arbib and Manes, 1974a), noting that $Q + X_0 \cong Q \times X_0$ in Vect, takes $X = - + X_0 = \hat{X}_0$, and represents a linear system with input space X_0, state space Q, and output space Y_0 in the form

$$\begin{aligned} &\tau: A \to Q, \qquad \text{the space of initial states is } \tau(A), \\ &\delta = (F, G): QX = Q + X_0 \to Q, \\ &\beta = H: Q \to Y_0. \end{aligned} \tag{2}$$

The decomposable system approach does not square well with the process transformation approach:

$$\begin{aligned} \delta&: QX \to Q, \\ \tau&: A \to Q, \\ \alpha&: A \to B, \\ \beta&: \hat{Q}X \to Y. \end{aligned} \tag{3}$$

When we take $X = \mathrm{id}$, (3) includes no representation of the input-dependent map $Q + A \to B$ one would look for in extending (1). However, translating (3) in the context of (2) we obtain

$$\begin{aligned} \delta = (F, G)&: Q + X_0 \to Q, \\ \tau&: A \to Q, \\ \alpha&: A \to B, \\ \beta&: Q + X_0 \to Y_0, \end{aligned} \tag{4}$$

where α now describes the recoding of initial states, and the representation $\beta: Q + X_0 \to Y_0$ is obtained on noting that, in Vect, $\hat{Q} \cong - + Q$, and we take $Y = - + Y_0$.

The crucial point in the above, then, is that we may identify $+$ with $\times$. This is a feature that Vect shares with any additive category (Arbib and Manes, 1975a, Section 5.2), and the following development is available in any additive category—in particular for the category R-Mod of modules over a commutative ring R. However, we restrict our attention to Vect for concreteness.

5. Definition. Let A, B, X_0, and Y_0 be vector spaces. Then a *linear system* is a restricted process transformation $M: (A, \hat{X}) \to (B, \hat{Y})$. More specifically, $M = (Q, F, G, \tau, \alpha, H, J)$, where

$(F, G): Q + X_0 \to Q$	is the *state dynamics*,
$\tau: A \to Q$	is the *initial state map*,
$\alpha: A \to B$	is the *initial throughput*,
$(H, J): Q + X_0 \to Y_0$	is the *output map*.

With any vector space A we may associate its countable copower

$$A^{\S} = \{(\ldots, a_n, \ldots, a_1, a_0) \mid \text{each } a_j \in A, \text{ only finitely many } a_j \text{ nonzero}\} \tag{6}$$

with the two associated maps

$$\begin{aligned} A\,\mathrm{in}_0&: A \to A^{\S}: a \mapsto (\ldots, 0, \ldots, 0, a), \\ Az&: A^{\S} \to A^{\S}: (\ldots, a_j, \ldots, a_1, a_0) \mapsto (\ldots, a_{j-1}, \ldots, a_0, 0), \end{aligned} \tag{7}$$

from which we may define

$$Ak = (z, \mathrm{in}_0)\colon A^{\S} + A \to A^{\S}. \tag{8}$$

We then have that the free X-dynamics over A, for $X = - + X_0$, is given by

$$\begin{aligned} &AX^{@} = A^{\S} + X_0^{\S}, \\ &A\mu_0 : AX^{@}X \to AX^{@} = Az + X_0k\colon A^{\S} + (X_0^{\S} + X_0) \to A^{\S} + X_0^{\S}, \\ &A\eta\colon A \to AX^{@} = A\,\mathrm{in}_0 + 0\colon A \to A^{\S} + X_0^{\S}. \end{aligned} \tag{9}$$

The reachability map $r\colon A^{\S} + X_0^{\S}$ is defined by the recursion

$$\begin{tikzcd} A \arrow[r, "\mathrm{in}_0"] \arrow[dr, "\tau"'] & A^{\S} + X_0^{\S} \arrow[d, "(r_A, r_X)"] & A^{\S} + X_0^{\S} + X_0 \arrow[l, "z+(z,\mathrm{in}_0)"'] \arrow[d, "(r_A, r_X)+X_0"] \\ & \underset{\sim}{Q} & \underset{\sim}{Q} + X_0 \arrow[l, "(F,G)"'] \end{tikzcd} \tag{10}$$

which unpacks as two simple recursions

$$\begin{tikzcd} A \arrow[r, "\mathrm{in}_0"] \arrow[dr, "\tau"'] & A^{\S} \arrow[d, "r_A"] & A^{\S} \arrow[l, "z"'] \arrow[d, "r_A"] \\ & \underset{\sim}{Q} & \underset{\sim}{Q} \arrow[l, "F"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_0 \arrow[r, "\mathrm{in}_0"] \arrow[dr, "G"'] & X_0^{\S} \arrow[d, "r_X"] & X_0^{\S} \arrow[l, "z"'] \arrow[d, "r_X"] \\ & \underset{\sim}{Q} & \underset{\sim}{Q} \arrow[l, "F"] \end{tikzcd} \tag{11}$$

yielding

$$r_A(\ldots, a_j, \ldots, a_1, a_0) = \sum_{j \geqslant 0} F^j \tau a_j\,; \qquad r_X(\ldots, x_j, \ldots, x_1, x_0) = \sum_{j \geqslant 0} F^j G x_j\,.$$

The crucial observation, which appears to be new, is *the complete symmetry in the treatment of A and X_0* in the reachability of the system. Setting X_0 to 0, we obtain $- + X_0 \simeq \mathrm{id}_{\mathbf{Vect}}$, and we recapture the decomposable machine setting for linear systems—but where we now realize that the input is better viewed (though the mathematical effect is the same) as a *continuing increment to the initial state*, added in anew at each time step. Setting A to 0 in (11), we recapture the "usual" model of a linear system in which the initial state is 0, and so there cannot be nonzero increments during the running of the system. These observations explain the somewhat anomalous position of decomposable systems within our general theory of machines in a category—as the one case in which the initial state $\tau\colon A \to \underset{\sim}{Q}$ is treated as an input map.

With this, we can now turn to computing the response of a linear process transformation with, in view of the above, special attention to the case $A =$

$B = 0$. In the present case, the general definition 3.15 of the response takes the form:

$$\begin{tikzcd}
A \arrow[r, "\mathrm{in}_0"] \arrow[d, "\alpha"'] & A^{\S} + X_0{}^{\S} \arrow[d, "\gamma"] & & A^{\S} + X_0{}^{\S} + X_0 \arrow[ll, "z+(z,\mathrm{in}_0)"'] \arrow[d, "\binom{\gamma}{r}+X_0"] \\
B \arrow[r, "\mathrm{in}_0"] & B^{\S} + Y_0{}^{\S} & B^{\S} + Y_0{}^{\S} + Y_0 \arrow[l, "z+(z,\mathrm{in}_0)"'] & B^{\S} + Y_0{}^{\S} + Q + X_0 \arrow[l, "B^{\S}+Y_0{}^{\S}+(H,J)"']
\end{tikzcd} \tag{12}$$

We may write

$$\gamma = \begin{pmatrix} \gamma_{BA} & \gamma_{BX_0} \\ \gamma_{Y_0A} & \gamma_{Y_0X_0} \end{pmatrix},$$

where $\gamma_{RS} : S^{\S} \to R^{\S}$ and (12) unpacks to yield the diagrams (13)–(16) below.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{in}_0"] \arrow[d, "\alpha"'] & A^{\S} \arrow[d, "\gamma_{BA}"] & A^{\S} \arrow[l, "z"'] \arrow[d, "\gamma_{BA}"] \\
B \arrow[r, "\mathrm{in}_0"] & B^{\S} & B^{\S} \arrow[l, "z"']
\end{tikzcd} \tag{13}$$

$$\gamma_{BA}(\ldots, a_j, \ldots, a_1, a_0) = (\ldots, \alpha(a_j), \ldots, \alpha(a_1), \alpha(a_0)).$$

This is a memoryless recording of the initial state symbols from A to B.

$$\begin{tikzcd}
A \arrow[r, "\mathrm{in}_0"] \arrow[dr, "0"'] & A^{\S} \arrow[d, "\gamma_{Y_0A}"] & A^{\S} \arrow[l, "z"'] \arrow[dl, "(z,\mathrm{in}_0H)\binom{\gamma_{Y_0A}}{r_A}"] \\
& Y_0{}^{\S} &
\end{tikzcd} \tag{14}$$

Setting $\gamma' = \gamma_{Y_0A}$, we have

$$\gamma'(\mathrm{in}_0\, a) = 0,$$
$$\gamma'(zw) = z\gamma'(w) + \mathrm{in}_0\, Hr_A(w).$$

Thus $\gamma'(\ldots, a_j, \ldots, a_1, a_0)_k = H \cdot r_A(\ldots, a_{j+k+1}, \ldots, a_{k+2}, a_{k+1})$ and records, with unit delay, the effect in Y_0, via H, of successive cumulative effects of the initial states.

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{in}_0"] \arrow[dr, "0"'] & X_0{}^{\S} \arrow[d, "\gamma_{BX_0}"] & X_0{}^{\S} \arrow[l, "z"'] \arrow[d, "\gamma_{BX_0}"] \\
& B^{\S} & B^{\S} \arrow[l, "z"']
\end{tikzcd} \tag{15}$$

which implies that $\gamma_{BX_0} = 0$—quite properly, since the inputs X_0 should not have any effect upon the initial state symbols B.

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{in}_0"] \arrow[dr, "\mathrm{in}_0 \cdot J"'] & X_0{}^{\S} \arrow[d, "\gamma_{Y_0X_0}"] & X_0{}^{\S} \arrow[l, "z"'] \arrow[dl, "(z, \mathrm{in}_0 H)\binom{\gamma_{Y_0X_0}}{r_X}"] \\ & Y_0{}^{\S} & \end{tikzcd} \tag{16}$$

Setting $\hat{\gamma} = \gamma_{Y_0X_0}$, we have

$$\hat{\gamma}(\mathrm{in}_0\, x) = \mathrm{in}_0\, Jx,$$

$$\hat{\gamma}(zw) = z\gamma'(w) + \mathrm{in}_0\, Hr_X(w),$$

so that

$$\hat{\gamma}(zw + \mathrm{in}_0\, x)_0 = Hr_X(w) + Jx \tag{17}$$

which is the sum of the contribution, via H, of the state $r_X(w)$ reached via previous X_0-inputs and the contribution, via J, of the present input x to the Y_0-output.

Modifying notation appropriately, we see that the $\hat{\gamma}$ of (17) is essentially the *result* f_M, in the sense of Eilenberg (1974, Section XVI.2), of the linear system of Definition 5 when A and B are restricted to be 0. Eilenberg associates with M the transformation from an input sequence

$$x = (x_0, x_0, \ldots, x_n, \ldots)$$

to both a state sequence

$$q = (0, q_1, \ldots, q_n, \ldots)$$

and an output sequence

$$y = (y_0, y_1, \ldots, y_j, \ldots)$$

given by the formulas

$$q_{n+1} = Fq_n + Gx_n,$$

$$y_n = Hq_n + Jx_n.$$

Then $f_M : X_0{}^{\mathbf{N}} \to Y_0{}^{\mathbf{N}}$ is the passage from x to y so defined, and we see that

$$f_M(x)_n = \hat{\gamma}(\ldots, 0, \ldots, x_0, x_1, \ldots, x_n).$$

To close the section, we specialize the definitions of series and parallel composition for restricted process transformations given in Section 5 to the case of linear systems with $A = B = 0$. Proposition 18 is obtained by specializing

Definition 5.14; Proposition 19 is obtained by specializing Definition 5.17 and taking $Y = Y_1 + Y_2 \cong Y_1 \times Y_2$, $B = B_1 \times B_2$, and letting c and h be the appropriate identities.

18. Proposition. *Given linear systems $M_1 = (Q_1, F_1, G_1, H_1, J_1): X_0 \to Y_0$ and $M_2 = (Q_2, F_2, G_2, H_2, J_2): Y_0 \to Z_0$, their serial connection $M = (Q, F, G, H, J): X_0 \to Z_0$ is defined by the equations*:

$$Q = Q_1 + Q_2,$$

$$\begin{tikzcd}
Q_1 + X_0 \arrow[r, "{(F_1,G_1)}"] & Q_1 \\
Q_1 + Q_2 + X_0 \arrow[u, "\mathrm{pr}_1 + X"] \arrow[r, "{(F,G)}"] \arrow[d, "\cong"'] & Q_1 + Q_2 \arrow[u, "\mathrm{pr}_1"'] \arrow[dd, "\mathrm{pr}_2"] \\
Q_2 + Q_1 + X_0 \arrow[d, "Q_2+(H_1,J_1)"'] & \\
Q_2 + Y_0 \arrow[r, "{(F_2,G_2)}"] & Q_2
\end{tikzcd}
\qquad
(F \mid G) = \left(\begin{array}{cc|c} F_1, & 0 & G_1 \\ G_2H_1, & F_2 & G_2J_1 \end{array}\right),$$

$$(H \mid J)\colon Q_1 + Q_2 + X_0 \xrightarrow{\cong} Q_2 + Q_1 + X_0 \xrightarrow{Q_2+(H_1,J_1)} Q_2 + Y_0 \xrightarrow{(H_2,J_2)} Z,$$

so that $(H \mid J) = (J_2H_1, H_2 \mid J_2J_1)$. $\blacksquare$

19. Proposition. *Given linear systems $M_1 = (Q_1, F_1, G_1, H_1, J_1): X_0 \to Y_1$ and $M_2 = (Q_2, F_2, G_2, H_2, J_2): X_0 \to Y_2$, their parallel connection $M = (Q, F, G, H, J): X_0 \to Y_1 \times Y_2$ is defined by the equations*

$$Q = Q_1 + Q_2,$$

$$\begin{tikzcd}
Q_1 + Q_2 + X_0 \arrow[r, "{(F,G)}"] \arrow[d, "\mathrm{pr}_i + X_0"'] & Q_1 + Q_2 \arrow[d, "\mathrm{pr}_i"] \\
Q_i + X_0 \arrow[r, "{(F_i,G_i)}"] & Q_i
\end{tikzcd}
\qquad
(F \mid G) = \left(\begin{array}{cc|c} F_1 & 0 & G_1 \\ 0 & F_2 & G_2 \end{array}\right),$$

$$\begin{tikzcd}
Q_1 + Q_2 + X_0 \arrow[r, "{(H,J)}"] \arrow[d, "\mathrm{pr}_i + X_0"'] & Y_1 + Y_2 \arrow[d, "\mathrm{pr}_i"] \\
Q_i + X_0 \arrow[r, "{(H_i,J_i)}"] & Y_i
\end{tikzcd}
\qquad
(H \mid J) = \left(\begin{array}{cc|c} H_1 & 0 & J_1 \\ 0 & H_2 & J_2 \end{array}\right). \quad \blacksquare$$

These do indeed coincide with the usual definitions of series and parallel composition of linear machines (see, e.g., Eilenberg, 1974, Sections 6, 7).

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